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THE RESISTANCE TO THE MOTION OF LOADS ALONG ELASTIC DIRECTIONS CAUSED BY THE RADIATION OF WAVES IN THEM†

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An analytic solution of the problem describing the oscillations of a string when a load moves on it with an arbitrary specified law of motion is obtained. Using the equation obtained, an analytic relationship is derived for the horizontal component of the total reaction force of the string acting at the point of application of the load.

WE WILL assume that a load P (a vertical constant force) moves along a string which lies on a viscous Winkler foundation. The small vertical oscillations of the string are described by the equation

$$u_{tt} - c^2 u_{xx} + 2\delta u_t + h^2 u(x,t) = 0 \tag{1}$$

where $c = \sqrt{(T/\rho)}$ is the velocity of propagation of the transverse waves in the string, T is the tension, ρ is the density per unit length, and the quantities δ and h represent its viscous and elastic properties, respectively. The motion of the load is given by the following relation

$$x = l(t) \in C_2[0,+\infty),\ \ l(0)=0,\ \ \dot{l}(0) = v_0 \geqslant 0,\ \ \ddot{l}(0) > 0$$

We will assume that the function $l(t)$ is a monotonically increasing function $l(+\infty) = +\infty$ and $\dot{l}(t) \neq c$.

The solution $u(x, t)$ will be sought separately for $x \gtrless 0$, i.e. $u(x, t) = u^{\pm}(x, t)$, $x \gtrless 0$, where the functions $u^{\pm}(x, t)$ are connected by the relations

$$u^+(0,t) = u^-(0,t),\ \ u_x^+(0,t) = u_x^-(0,t),\ \ t > 0$$

(by the value of the function we mean the corresponding single-sided limit). As we know [1], the solution

$$u^+(x,t) = \begin{cases} u_2(x,t), & x > l(t) \ \ ((x,t) \in D_2) \\ u_1(x,t), & x < l(t) \ \ ((x,t) \in D_1),\ \ x > 0, t > 0 \end{cases}$$

must then satisfy the matching conditions

$$[u(x,t)]|_{x=l(t)} = 0,\ \rho(c^2 - \dot{l}^2(t))[u_x(x,t)]|_{x=l(t)} = -P$$
$$([u(x,t)]|_{x=l(t)} \overset{\text{def}}{\equiv} u_2(l(t),t) - u_1(l(t),t)) \tag{2}$$

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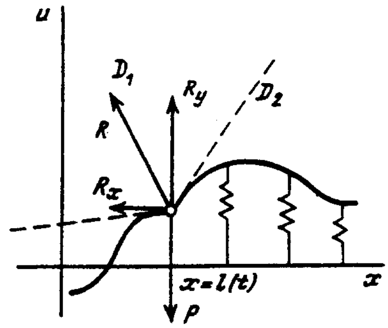

FIG. 1.

Then the horizontal component of the total reaction R of the string at the point $x=l(t)$ (see Fig. 1) is given by the equation [1]

$$R_x = -\tfrac{1}{2}\rho(c^2 - \dot{l}^2(t))[u_x^2(x,t)]|_{x=l(t)} \tag{3}$$

We will choose the initial conditions in the form

$$\begin{aligned} &u^{\pm}(x,0) = \varphi^{\pm}(x) = (2\gamma\ T)^{-1}P_0 e^{\mp\gamma x} (\gamma = h/c) \\ &u_t^{\pm}(x,0) = 0,\ x \gtrless 0 \end{aligned} \tag{4}$$

Note than when $P_0 = P$ the load "takes off" from a position corresponding to the profile of the string for a stationary load P; if $P_0 = 0$ we obtain an undisturbed string (zero initial conditions). We will seek a solution of the problem bounded at infinity $((x\ t) \to \infty)$.

We will apply a Laplace transformation separately for $x \gtrless 0$ to Eq. (1), where

$$U^+(x,p) = L\{u^+(x,t)\} = \int_0^{l_1(x)} u_2 e^{-pt} dt + \int_{l_1(x)}^{+\infty} u_1 e^{-pt} dt \tag{5}$$

and $l_1(x)$ is the inverse function to $l(t)$. Then, taking conditions (2) into account we have

$$L\{u_{tt}\} = p^2 L\{u^+(x,t)\} - pu^+(x,0) + e^{-pl_1(x)}[u_t(x,t)]|_{t=l_1(x)}$$

or, by virtue of the relation $[u_t]|_{x=l(t)} = -\dot{l}(t)[u_t]|_{x=l_1(x)}$ (obtained by differentiating the first relation in (2)) we find

$$u_{tt}^+ \doteqdot p^2 U^+(x,p) - p\varphi^+(x) + \frac{P\dot{l}(t)}{\rho(c^2 - \dot{l}^2(t))} e^{-pl_1(x)},\ t = l_1(x)$$

The relation between U_{xx}^+ and $L\{u_{xx}^+\}$ is calculated by differentiating (5).

We finally obtain an operator representation of problem (1), (2) and (4)

$$\begin{aligned} &U_{xx}^{\pm} - \lambda^2 U^{\pm}(x,p) = -PT^{-1}\dot{l}_1(x)e^{-pl_1(x)}\eta(x) - \\ &-c^{-2}\varphi^{\pm}(x)(p+2\delta);\ \lambda = c^{-1}(p^2 + 2\delta p + h^2)^{1/2} \end{aligned} \tag{6}$$

where $\eta(x)$ is the unit function, and $\lambda > 0$ when $p > 0$ [2].

Two of the four arbitrary constants from the solutions of Eqs (6) will be assumed to be zero, taking into account the steady-state initial conditions (4) (i.e. there are no "backward" waves) and the condition at infinity. The two remaining constants are found from the "matching" conditions at the point $x = 0$

$$U^+(0,p)=U^-(0,p),\ \ U_x^+(0,p)=U_x^-(0,p)\ \ (\operatorname{Re} p>0)$$

As a result we uniquely find the image of the required solution

$$U^-(x,p)=-\frac{P_0e^{\lambda x}}{2Tp\lambda}+\frac{Pe^{\lambda x}}{2T\lambda}F(0,p)+\frac{1}{p}\varphi^-(x)$$

$$U^+(x,p)=\frac{P}{2T\lambda}\{e^{\lambda x}F(x,p)+e^{-\lambda x}\int_0^x \dot{l}_1(\xi)e^{-pl_1(\xi)+\lambda\xi}\,d\xi\}-$$

$$-\frac{P_0e^{-\lambda x}}{2Tp\lambda}+\frac{1}{p}\varphi^+(x),\ \ F(x,p)=\int_x^{+\infty}\dot{l}_1(\xi)e^{-pl_1(\xi)-\lambda\xi}d\xi,\ \ \operatorname{Re} p>0 \tag{7}$$

A numerical calculation, and especially an analysis of the solution $U^{\pm}(x, p)$ obtained, and particularly its partial derivatives when calculating the inversion formula $u^{\pm}=L^{-1}\{U^{\pm}\}$ directly from (7), is fairly difficult (particularly from the point of view of obtaining the required accuracy) in view of the oscillations of the integrals. Hence, to achieve the inversion an investigation is made of the asymptotic behaviour of the integrals (7). In view of the complexity of the expressions obtained and the multitude of cases depending on the ratio between h and δ, we will confine ourselves here to the case when $\delta=0$.

Consider the function $F(x, p)$, regular in the half-plane $\operatorname{Re} p>0$ when $\lambda=c^{-1}\sqrt{(p^2+h^2)}$, $x>0$. The investigation of its asymptotic behaviour when $p\to\infty$, $\operatorname{Re} p\geqslant\alpha>0$ can be reduced, in principle, to a consideration of the standard Laplace integral [3], where the point of maximum contribution $\xi=x$ if the quantity λ is expanded in a Laurent series in the neighbourhood of the point $p=\infty$, and appropriate estimates are made. However, in this case this is more simply achieved by integrating the expression for $F(x,p)$ by parts and estimating the complex expressions obtained taking into account the smoothness of $l(t)$ in the general scheme for constructing an asymptotic series.

As a result we obtain that

$$F(x,p)=e^{-pl_1(x)-\lambda x}\left(\frac{\dot{l}_1(x)}{\dot{l}_1(x)+c^{-1}}\frac{1}{p}+O(\frac{1}{p^2})\right)$$

$$p\to\infty,\ \ \operatorname{Re} p\geqslant\alpha>0,\ \ x>0$$

From these estimates and a similar estimate for the second term it follows, in particular, that $U^+(x, p)$ is the image (since the sufficient conditions [2] are satisfied), i.e. taking into account the inversion formula the quantity $u(x, t)$ obtained is a solution of the problem. We then apply the inversion formula to the term in question, in which part of the integral $F(x, p)$ (namely, the integral in the limits from $x_1{}^*$ to ∞), where $x_1{}^*$ is the first positive root of the equation

$$c^{-1}x+t=F_1(x_1^*)\overset{\text{def}}{\equiv} l_1(x_1^*)+c^{-1}x_1^*$$

makes a zero contribution to the quantity $u^+(x, t)$ on the basis of Jordan's lemma. Inversion of the remaining part of $F(x,p)$ can be carried out using the well-known operational formulae [4], if we change the order of integration in the inversion formula. The remaining terms from (7) are inverted using exactly the same scheme, except, for example, that in the next term one must take into account the fact that for a velocity $\dot{l}(t)$ smaller or greater than the critical value, i.e. c, the point of maximum contribution will be the beginning or end of the integration section, respectively. Finally, when $\dot{l}(t)<c$ we have

$$u^+(x,t)=A(x,t,x_1^*,x)\eta(t-l_1(x))+A(x,t,x_2^*,0)\eta(t-c^{-1}x)+$$

$$+\varphi^+(x)+A_0(x,t,c^{-1}x)\eta(t-c^{-1}x)$$

$$u^-(x,t)=A(x,t,x_3^*,0)\eta(t+c^{-1}x)+\varphi^-(x)+A_0(x,t,-c^{-1}x)\eta(t+c^{-1}x)$$

$$\varsigma_1(\xi,x,t)=h[(t-l_1(\xi))^2+c^{-2}(\xi-x)^2]^{1/2}$$

$$\varsigma_2(\xi,x)=h[\xi^2-c^{-2}x^2]^{1/2}$$

$$A(x,t,\beta,\alpha)=\frac{Pc}{2T}\int_\alpha^\beta \dot{l}_1(\xi)J_0(\varsigma_1(\xi,x,t))d\xi$$

$$A_0(x,t,\alpha)=-\frac{P_0c}{2T}\int_\alpha^t J_0(\xi_2(\xi,x))d\xi \tag{8}$$

($J_0(x)$ is a Bessel function of zero order).

If $\dot{l}(t)>c$, then in the second term in the formula for $u^+(x, t)$ the integration must be carried out in the limits from $x_2{}^*$ to x, while $\eta(t-c^{-1}x)$ is replaced by $\eta(t-l_1(x))$.

In Eqs (8), if $t<l_1(x)$, i.e. the load "does not reach" the point x, then $x_2{}^*$ is the first positive root of the equation $t-c^{-1}x=l_1(x_2{}^*)-c^{-1}x_2{}^*$, if $t\geqslant l_1(x)$, then $x_2{}^*=x$. If the load moves with a velocity greater than the critical velocity, then when $l_1(x\leqslant t\leqslant c^{-1}x)$ the quantity $x_2{}^*$ is found from the previous equation, and $x_2{}^*=0$ when $t\geqslant c^{-1}x$. The quantity $x_3{}^*$ is the first positive root of the equation

$$t+c^{-1}x=l_1(x_3^*)+c^{-1}x_3^*,\quad x<0$$

Representation (8) enables us to understand the structure of the solution $u(x,t)$. The term $A_1=A(x, t, x_1{}^*, x)\eta(t-l_1(x))$ from (8) is a wave radiated backward by the load, and hence the contribution to the quantity A_1 at the first point x and at a given instant of time t introduces points of the section $\xi\in[x, x_1{}^*]$ (the section of integration), already excited by the load P, the excitation from which is able to arrive at the given point x. The term $A_2=A(x, t, x_2{}^*, 0)\eta(t-c^{-1}x)$ corresponds to a wave radiated forward by the load, which explains the difference in the sections of integration for the cases $\dot{l}(t)\gtreqless c$. The next two terms $u^+(x, t)$ are connected with the initial position of the string.

The calculation of the force of resistance to the motion of the load, i.e. the quantity R_x given by (3), which is calculated from the formula

$$R_x={}^1/_2P[(u_x)_2(x,l_1(x))+(u_x)_1(x,l_1(x))]$$

is facilitated by the physical meaning of the terms. In the case of a subcritical velocity, it is then necessary to take into account that the quantities $\partial A_1/\partial x$ and $\partial A_2/\partial x$ on the line $t=l_1(x)$ are discontinuous, and $x_{1,2}{}^*$ are functions x,t.

After simple but lengthy calculations and estimates we finally obtain

$$R_x(x)=\frac{P^2h^2}{2Tc}\int_0^x \dot{l}_1(\xi)\frac{J_1(\varsigma_1(\xi,x,l_1(x)))}{\varsigma_1(\xi,x,l_1(x))}(x-\xi)d\xi+$$
$$+\frac{P^2}{2T}\frac{c\dot{l}_1(x)}{1-c^2\dot{l}_1^2(x)}-\frac{PP_0h^2}{2Tc}x\int_{x/c}^{l_1(x)}\frac{J_1(\varsigma_2(\xi,x))}{\varsigma_2(\xi,x)}d\xi+$$
$$+\frac{P_0P}{2T}(1-e^{-\gamma x}),\quad x>0$$

Here $J_1(x)$ is a function of the first order.

In particular, we find the reactive force of the string at the initial instant (at the instant of "starting")

$$R_x(x=0)=-\frac{P^2}{2T}\frac{cv_0}{c^2-v_0^2}$$

A calculation of $R_x(x)$ using these equations shows that, for example, when $\dot{l}(t)=l_0=\text{const},\ t\geqslant t_0>0$, the value of R_x oscillates with a negative mean (when integration is carried out over a fairly large interval) and the frequency, characteristic for this elastic system (for $t\geqslant t_0$), also approaches zero as $x\to+\infty$.

When the velocity $\dot{l}(t)>c$, the value of the jump $[u_x]|_{t=l_1(x)}$ is given by the quantity

$$r(x,t)=A(x,t,x_1^*,x)\eta(t-l_1(x))+A(x,t,x,x_2^*)\eta(t-l_1(x))$$

which corresponds to a perturbation which follows after the load; then the following relations hold

$$\frac{\partial r_1}{\partial x}(l(t),t)=-\frac{P}{T}\frac{c^2}{\dot{l}^2(t)-c^2},\quad \frac{\partial r_2}{\partial x}(l(t),t)=0 \tag{9}$$

The value of the jump in the derivative, given by (9), obviously satisfies condition (2). In this case

$$R_x(x)=-\frac{P^2}{2T}\frac{c^2}{\dot{l}^2(t)-c^2}-\frac{P_0P}{2T}e^{-\gamma x},\quad t=l_1(x)$$

Note that problems of passing through the critical velocity in the case of uniformly accelerated motion were investigated previously in [5]. The method of obtaining the asymptotic form of the solutions, an analysis of these and a number of important applications were considered in [6].

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